Noncommutative differential calculus and lattice gauge theory

1993 J. Phys. A: Math. Gen. 26 1927

(http://iopscience.iop.org/0305-4470/26/8/019)

J. Phys. A: Math. Gen. 26 (1993) 1927–1949. Printed in the UK

Non-commutative differential calculus and lattice gauge theory

A Dimakis, F Müller-Hoissen and T Striker
Institut für Theoretische Physik, Bunsenstrasse 9, Göttingen, Federal Republic of Germany

Received 11 November 1992

Abstract. We study consistent deformations of the classical differential calculus on algebras of functions (and, more generally, commutative algebras) such that differentials and functions satisfy non-trivial commutation relations. For a class of such calculi it is shown that the deformation parameters correspond to the spacings of a lattice. These differential calculi *generate* a lattice on a space continuum. The whole setting of a lattice theory can then be deduced from the continuum theory via deformation of the standard differential calculus. In this framework one just has to express the Lagrangian for the continuum theory in terms of differential forms. This expression then also makes sense for the deformed differential calculus. There is a natural integral associated with the latter. Integration of the Lagrangian over a space continuum then produces the correct lattice action for a large class of theories. This is explicitly shown for the scalar field action and the action for SU(m) gauge theory.

1. Introduction

Non-commutative geometry deals with differential calculus on algebras which are, in general, not commutative [1]. One of the ideas used to motivate the study of non-commutative geometry in physics is to remove ultraviolet divergencies in quantum field theories by replacing coordinates by non-commuting operators (see [2–4], for example). The present work shows that such a regularization can be achieved by keeping coordinates commutative, and merely deforming the differential calculus in such a way that coordinates (or, more generally, functions) no longer commute with differentials. The differential calculus which we discuss in sections 3–6 is a deformation of the classical differential calculus in this sense. We find that via this deformation a continuum theory is replaced by a corresponding lattice theory. Lattice formulations are amongst the most frequently investigated regularizations of quantum field theories, in particular if non-perturbative effects are addressed (see [5–9], for example).

In section 2 we introduce a more general class of differential calculi on commutative algebras and discuss consistency conditions. Section 3 deals with the simplest case of only one dimension. In section 4 we formulate gauge theory in the one-dimensional case and establish some basic relations with lattice gauge theory. Many features of our formalism are already present in one dimension and the mathematical formulae derived in this case have obvious generalizations to dimensions greater than one. Therefore, and also because of pedagogical reasons,

0305-4470/93/081927+23$07.50

the one-dimensional case is discussed first in sections 3 and 4. Sections 5 and 6 then generalize the results to higher dimensions. For a scalar field and for a gauge field we derive the lattice actions from the corresponding continuum actions. In order to do this we need an integral for the deformed differential calculus. This is constructed in appendix A.

A generalized Laplace–Beltrami operator is defined in appendix B. We show that it coincides with the known lattice version of the classical operator.

In appendix C we present a classification of all consistent differential calculi with real constant coefficients in *two* dimensions. Some properties of new consistent calculi are discussed in appendix D. Section 7 contains our conclusions.

2. Differential calculus and consistency conditions

In the ordinary differential calculus on manifolds differentials commute with functions, i.e.

$$[\,x^i\,,\,\mathrm{d}x^j\,] = 0 \qquad (i,j = 1,\ldots,n) \tag{2.1}$$

in terms of (real) coordinates x^i. In this work we consider deformations of (2.1) of the form

$$[\,x^i\,,\,x^j\,] = 0 \tag{2.2}$$

$$[\,x^i\,,\,\mathrm{d}x^j\,] = \sum_{k=1}^{n} \mathrm{d}x^k\, C^{ij}{}_k \tag{2.3}$$

where the $C^{ij}{}_k$ are (complex) constants which are constrained by the requirement of a consistent differential calculus. An example with interesting relations to physics has already been studied in [11].

Let us first recall the notion of a *differential calculus* on an algebra $\mathcal{A}$ [1, 10]. This is a $\mathbb{Z}$-graded algebra

$$\wedge(\mathcal{A}) = \bigoplus_{r\in\mathbb{Z}} \overset{r}{\wedge}(\mathcal{A}) \qquad \overset{0}{\wedge}(\mathcal{A}) = \mathcal{A} \qquad \overset{r}{\wedge}(\mathcal{A}) = \{0\} \qquad \forall r < 0\,.$$

The elements of $\wedge^r(\mathcal{A})$ are called r-forms. There is a $\mathbb{C}$-linear *exterior derivative* operator $\mathrm{d} : \wedge^r(\mathcal{A}) \to \wedge^{r+1}(\mathcal{A})$ which satisfies

$$\mathrm{d}^2 = 0 \tag{2.4}$$

$$\mathrm{d}(\omega\,\omega') = (\mathrm{d}\omega)\,\omega' + (-1)^r\,\omega\,\mathrm{d}\omega' \tag{2.5}$$

where ω and ω' are r- and r'-forms, respectively.

In our case $\mathcal{A}$ is the commutative algebra generated by the coordinate functions x^i. Restrictions on the constants $C^{ij}{}_k$ arise from the following procedures.

(i) Applying d to (2.2) and using (2.3) we find

$$C^{ij}{}_k = C^{ji}{}_k\,. \tag{2.6}$$

(ii) Commuting $\mathrm{d}x^i$ through (2.2) leads to

$$\sum_{\ell=1}^{n} C^{ik}{}_{\ell}\, C^{j\ell}{}_{m} = \sum_{\ell=1}^{n} C^{jk}{}_{\ell}\, C^{i\ell}{}_{m} \tag{2.7}$$

respectively

$$\sum_{\ell=1}^{n} C^{k[i}{}_{\ell}\, C^{j]\ell}{}_{m} = 0 \tag{2.8}$$

by using (2.6). This means that the n matrices $\mathbf{C}^i$ with entries $C^{ik}{}_{\ell}$ mutually commute.
(iii) Commuting x^k through (2.3) also yields (2.7) and therefore no additional conditions.
(iv) Acting with d on (2.3) and using (2.5) enforces the classical commutation rule

$$\mathrm{d}x^i\,\mathrm{d}x^j = -\mathrm{d}x^j\,\mathrm{d}x^i \tag{2.9}$$

for the differentials. The equations obtained by commuting x^k through these relations are identically satisfied.

Remark. (2.2), (2.3) and (2.9) define an algorithm to *order* polynomials in x^i and $\mathrm{d}x^j$, e.g. to write them as linear combinations of the monomials $x^{i_1}\cdots x^{i_r}\mathrm{d}x^{j_1}\cdots\mathrm{d}x^{j_s}$ with $i_1 \leqslant i_2 \leqslant \cdots \leqslant i_r$ and $j_1 < j_2 < \cdots < j_s \leqslant n$. The ordering relations (2.2), (2.3) and (2.9) can be applied, however, in various sequences and may lead to different results. If (2.6) and (2.7) are satisfied, different ways of ordering lead to the same result. It is sufficient to check this for cubic monomials (cf [12, section 5.6]).

We have two natural ways to extend the conjugation * of complex numbers to the differential algebra $\wedge(\mathcal{A})$. Besides demanding that * commutes with d and that ** is the identity, we have the choice between the following two further assumptions:

(a) * is an automorphism of $\wedge(\mathcal{A})$. Then (2.3) requires the $C^{ij}{}_{k}$ to be real.
(b) * is a graded anti-automorphism

$$(\omega\,\omega')^* = (-1)^{rr'}\,\omega'^*\,\omega^* \tag{2.10}$$

for all differential forms ω, ω' of grades r and r', respectively. The $C^{ij}{}_{k}$ then have to be imaginary as a consequence of (2.3).

In this work we make the first assumption (a). Only in this case we can have real coefficients in (2.3) which is essential for the applications in the following sections.

There are examples of (deformed) differential calculi on (non-commutative) algebras $\mathcal{A}$ in the literature (see [10, 13], for example) for which the exterior derivative operator d can be expressed as a commutator with a certain 1-form ϑ

$$\mathrm{d}f = [\,\vartheta\,,\,f\,] \qquad \forall f \in \mathcal{A} \tag{2.11}$$

and, more generally,

$$\mathrm{d}\omega = [\,\vartheta\,,\,\omega\,]_{(r)} \tag{2.12}$$

where ω is an r-form and $[\ ,\ ]_{(r)}$ is the commutator or anticommutator depending on whether r is even or odd. The Leibniz rule (2.5) is then guaranteed and $d^2 = 0$ imposes the condition

$$[\,\vartheta^2\,,\,\omega\,] = 0 \qquad \forall\omega \tag{2.13}$$

on ϑ. Let us write ϑ as

$$\vartheta = \sum_{i=1}^{n} \mathrm{d}x^i\, b_i(x) \tag{2.14}$$

with functions b_i. From (2.11) with f replaced by x^i we find the following necessary condition for the existence of such a 1-form ϑ in case of the differential calculi under consideration

$$\sum_{i=1}^{n} \mathbf{C}^i\, b_i = -\mathbf{1} \tag{2.15}$$

where $\mathbf{1}$ is the $n \times n$ unit matrix. For the standard differential calculus (i.e. $\mathbf{C}^i = 0$) there is no ϑ. If such a ϑ exists for a deformed differential calculus, this can considerably simplify calculations. If there is no ϑ it may be possible to enlarge the original algebra in such a way that a ϑ exists in the larger differential algebra (cf [10] for the case of quantum groups).

3. Differential calculus in one dimension

The one-dimensional case with only a single coordinate x already exhibits the essential features, in particular the origin of the connection between non-commutative differential calculus and lattice theory.

The commutation relations between coordinates and differentials are

$$[\,x\,,\,\mathrm{d}x\,] = \mathrm{d}x\, a \tag{3.1}$$

with an arbitrary real† constant a. This leads to a consistent differential calculus. As a consequence of (3.1) we have

$$x^n\,\mathrm{d}x = \mathrm{d}x\,(x + a)^n \tag{3.2}$$

and therefore

$$f(x)\,\mathrm{d}x = \mathrm{d}x\, f(x + a) \tag{3.3}$$

for a function f. Commuting $f(x)$ through $\mathrm{d}x$ has the effect of a translation of the argument by a. The differential calculus with $a \neq 0$ thus simulates the action of a discrete translation group.

† Mathematically we may allow a complex constant a. But our interpretation of the calculus is bound to real a.

Let us introduce a *right partial derivative* by

$$\mathrm{d}f = \mathrm{d}x\,(\vec{\partial}\,f)(x)\,. \tag{3.4}$$

From the Leibniz rule for d we find that $\vec{\partial}$ has to satisfy the modified Leibniz rule

$$(\vec{\partial}\,fh)(x) = (\vec{\partial}\,f)(x)\,h(x) + f(x+a)\,(\vec{\partial}\,h)(x) \tag{3.5}$$

and we find

$$(\vec{\partial}\,f)(x) = \frac{1}{a}\,[\,f(x+a) - f(x)\,]\,. \tag{3.6}$$

Surprisingly, $\vec{\partial}$ is just the *discrete derivative*. In the limit $a \to 0$ it becomes the ordinary partial derivative (on differentiable functions). It follows that a *'constant' function* in the sense that $\mathrm{d}f = 0\ \forall x$ is just a periodic function with period a.

If f is an arbitrary function of x, we can use $[f(x), x] = 0$, the Leibniz rule for d and (3.1) to calculate the commutator of f with $\mathrm{d}x$

$$[\,f(x)\,,\mathrm{d}x\,] = [\,x\,,\mathrm{d}f(x)\,] = [\,x\,,\mathrm{d}x\,]\,(\vec{\partial}\,f)(x) = \mathrm{d}x\,a\,(\vec{\partial}\,f)(x) = a\,\mathrm{d}f\,. \tag{3.7}$$

We also have a *left partial derivative* defined by

$$\mathrm{d}f = (\overleftarrow{\partial}\,f)(x)\,\mathrm{d}x\,. \tag{3.8}$$

The corresponding modified Leibniz rule is

$$(\overleftarrow{\partial}\,fh)(x) = (\overleftarrow{\partial}\,f)(x)\,h(x) + f(x-a)\,(\overleftarrow{\partial}\,h)(x)\,. \tag{3.9}$$

Hence

$$(\overleftarrow{\partial}\,f)(x) = \frac{1}{a}\,[\,f(x) - f(x-a)\,]\,. \tag{3.10}$$

Comparing this with (3.6) yields

$$(\overleftarrow{\partial}\,f)(x) = (\vec{\partial}\,f)(x-a)\,. \tag{3.11}$$

Remark 1. The differential calculus with the constant a replaced by an arbitrary function $a(x)$ is also consistent and all the formulae above remain valid. However, it can be (at least formally) transformed into the calculus with constant a. This is seen as follows.

Let y be a function of x. Then

$$\begin{aligned}[\,y\,,\mathrm{d}y\,] &= [\,y\,,\mathrm{d}x\,]\,(\vec{\partial}\,y)(x) = [\,x\,,\mathrm{d}y\,]\,(\vec{\partial}\,y)(x)\\ &= [\,x\,,\mathrm{d}x\,]\,(\vec{\partial}\,y)^2 = \mathrm{d}x\,a\,(\vec{\partial}\,y)^2 = \mathrm{d}y\,a\,(\vec{\partial}\,y)(x)\end{aligned} \tag{3.12}$$

using (3.4) and (3.1). This shows that

$$[\,y\,,\mathrm{d}y\,] = \mathrm{d}y\,a(y) \tag{3.13}$$

can be transformed to (3.1) if we can solve the equation $(\vec{\partial}\,y)(x) = a(y)/a$. Indeed, the formal expression $x = a\int \mathrm{d}y\,a(y)^{-1}$ does the job (see appendix A for the definition of the integral).

Remark 2. The main physical point in this article is the use of non-commutative differential calculus as a bridge between continuum and lattice theories. It is then natural to choose the x^i in (2.2) and (2.3) as *functions* on $\mathbb{R}^n$ (or some n-dimensional manifold). There are, however, many other representations to which the formalism applies equally well. As an example, the $(2m+1)\times(2m+1)$ matrix

$$x = \frac{1}{2m}\,\mathrm{diag}(c - ma, c-(m-1)a, \ldots, a, \ldots, c + ma) \tag{3.14}$$

(where c is a constant) generates a commutative algebra by matrix multiplication. Together with

$$\mathrm{d}x = \begin{pmatrix} 0 & \ldots & \ldots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & & \ddots & \vdots \\ 1 & 0 & \ldots & 0 \end{pmatrix} \tag{3.15}$$

it satisfies (3.1) and $(\mathrm{d}x)^2 = 0$. The exterior derivative d is represented by (2.12) with $\vartheta = -(1/a)\,\mathrm{d}x$.

4. Gauge fields in one dimension

In spite of the non-commutativity between functions and differentials it is possible to formulate gauge theory in the standard way. In the following this is discussed for only one dimension. The generalization to higher dimensions is straightforward and essentially only adds indices to the relevant quantities (see section 6). Let

$$\mathbf{A} = \mathrm{d}x\, A(x) \tag{4.1}$$

be a matrix-valued 1-form which transforms like a connection according to the usual gauge transformation law

$$\mathbf{A}' = U\,\mathbf{A}\,U^{-1} - \mathrm{d}U\,U^{-1} \tag{4.2}$$

where $U(x)$ is a function with values in a matrix group. For $a^i \neq 0$ the last term in (4.2) involves a *finite* difference of group elements. The connection component $A(x)$ is therefore not Lie algebra but rather group algebra valued. From the transformation formula (4.2) we deduce

$$U(x+a)\,A(x) - A'(x)\,U(x) = (\vec{\partial}\,U)(x) = \frac{1}{a}\,[\,U(x+a) - U(x)\,] \tag{4.3}$$

which implies that

$$G(x) := \mathbf{1} - a\,A(x) \tag{4.4}$$

transforms according to

$$G'(x) = U(x+a)\,G(x)\,U(x)^{-1}\,. \tag{4.5}$$

If ψ transforms according to $\psi' = U\,\psi$ we can define a covariant derivative

$$D\psi := \mathrm{d}\psi + \mathbf{A}\,\psi \tag{4.6}$$

which as a consequence of (4.2) satisfies the same transformation law as ψ. The *left covariant derivative* of ψ defined by

$$D\psi := (\overleftarrow{\nabla}\,\psi)(x)\,\mathrm{d}x \tag{4.7}$$

then also transforms like ψ and is explicitly given by

$$\begin{aligned}(\overleftarrow{\nabla}\,\psi)(x) &= (\overleftarrow{\partial}\,\psi)(x) + A(x-a)\,\psi(x-a)\\ &= \frac{1}{a}\,[\,\psi(x) - G(x-a)\,\psi(x-a)\,]\,.\end{aligned} \tag{4.8}$$

If we want to read off covariant *right* components from (4.6) we are faced with the problem that the differential $\mathrm{d}x$ is not invariant under the adjoint action of the gauge group. However

$$Dx := \mathrm{d}x - a\,\mathbf{A} = \mathrm{d}x\,G(x) \tag{4.9}$$

is covariant, i.e.

$$D'x := U(x)\,Dx\,U(x)^{-1}\,. \tag{4.10}$$

In the following we will make the assumption that $G(x)$ is an element of the gauge group and therefore invertible. This is consistent with the homogeneous transformation law (4.5). As a consequence of (4.4) the connection component $A(x)$ is then an element of the group algebra. Defining a *right covariant derivative* by

$$D\psi := Dx\,(\overrightarrow{\nabla}\,\psi)(x) \tag{4.11}$$

we can conclude that

$$\begin{aligned}(\overrightarrow{\nabla}\,\psi)(x) &= G(x)^{-1}\,[\,(\overrightarrow{\partial}\,\psi)(x) + A(x)\,\psi(x)\,]\\ &= \frac{1}{a}\,[\,G(x)^{-1}\,\psi(x+a) - \psi(x)]\end{aligned} \tag{4.12}$$

transforms covariantly. The *covariant differences* which appear in (4.8) and (4.12) are familiar expressions in lattice gauge theory (see [6], for example).

As a consequence we have

$$(\overleftarrow{\nabla}\,\psi)(x) = G(x-a)\,(\overrightarrow{\nabla}\,\psi)(x-a)\,. \tag{4.13}$$

Remark. The equation

$$D\psi = 0 \tag{4.14}$$

is equivalent to

$$\psi(x+a) = G(x)\,\psi(x) \tag{4.15}$$

and thus generalizes the periodicity condition. Using this fact the boundary conditions for the quenched or twisted Eguchi–Kawai models [14, 15] can be formulated in an elegant way.

5. Generalization to higher dimensions

In this section we specify a differential calculus in n dimensions by†

$$[\, x^i \,, \mathrm{d}x^j \,] = a^i \, \delta^{ij} \, \mathrm{d}x^j \qquad \text{(no summation)} \tag{5.1}$$

with real constants a^i. This is just the differential calculus previously considered in each dimension. It is therefore consistent and the formulae which we have in the one-dimensional case generalize in an obvious way. By rescaling of the coordinates x^i we can achieve the condition that all non-vanishing a^i are equal and positive. An advantage of distinguishing the a's corresponding to different dimensions is the possibility of taking the limit $a^i \to 0$ for each i separately.

Left and right partial derivatives are introduced by‡

$$\mathrm{d}f(x) = \sum_{i=1}^{n} (\partial_{-i} f)(x) \, \mathrm{d}x^i = \sum_{i=1}^{n} \mathrm{d}x^i \, (\partial_i f)(x) \tag{5.2}$$

and we find

$$(\partial_i f)(x) = \frac{1}{a^i} \, [\, f(x + a^i) - f(x) \,] \tag{5.3}$$

$$(\partial_{-i} f)(x) = \frac{1}{a^i} \, [\, f(x) - f(x - a^i) \,] \tag{5.4}$$

where we use the notation

$$(x + a^i)^j := x^j + \delta^{ij} \, a^i \, . \tag{5.5}$$

According to (2.9) the differentials $\mathrm{d}x^i$ anticommute as in the standard differential calculus. This allows us to define a *Hodge* $\star$ *operator* for the Euclidean§ metric (which in the coordinates x^i has the components δ_{ij}) in the familiar way on products of differentials by

$$\star \, (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k}) := \frac{1}{(n-k)!} \sum_{i_{k+1},\dots,i_n} \epsilon^{i_1 \dots i_k}{}_{i_{k+1} \dots i_n} \, \mathrm{d}x^{i_{k+1}} \cdots \mathrm{d}x^{i_n} \tag{5.6}$$

where $\epsilon_{i_1 \dots i_n}$ is the totally antisymmetric Levi–Civita symbol. It then satisfies the familiar rules like (B.3) and (B.4) in appendix B. But because of the non-commutativity between differentials and functions, the operator $\star$ no longer commutes with functions. We define

$$\star \, (\, f(x) \, \omega \,) := (\star \omega) \, f(x) \tag{5.7}$$

† As it stands this equation only holds in a specific coordinate system and thus breaks general covariance. Equation (2.3), however, is invariant under coordinate transformations if $C^{ij}{}_k$ transforms like a tensor.

‡ The minus sign in front of an index takes the role of the leftarrow superscript used in the one-dimensional case.

§ We can as well consider the Minkowski metric $(\eta_{ij}) = \mathrm{diag}(-1, 1, \dots, 1)$. One may think of identifying δ^{ij} in (5.1) with the space-time metric. Note, however, that by reversing the sign of one of the coordinates x^i the corresponding eigenvalue of δ^{ij} will be replaced by -1 (if we do not change the sign of the respective a^i). The δ^{ij} in (5.1) is therefore not necessarily related to the space-time metric.

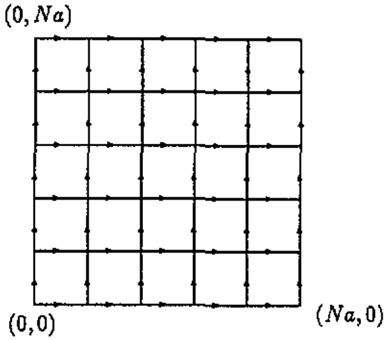

Figure 1. A rectangular $N \times N$ lattice with spacing a. The arrows indicate the way in which the discrete derivatives of ϕ are calculated in (5.12).

for functions f and differential forms ω. As a consequence we have

$$\star\,(\mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_k}\,f(x)\,) = \star(\mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_k})\,f(x-a^{i_1}-\cdots-a^{i_k})\,. \tag{5.8}$$

In the continuum theory the (Euclidean) Lagrangian for a real scalar field ϕ is

$$\mathcal{L} := \tfrac{1}{2}(-1)^{n-1}(\star\mathrm{d}\phi)\,\mathrm{d}\phi + \tfrac{1}{2}\,m^2(\star\phi)\,\phi \tag{5.9}$$

in terms of ordinary differential forms. Let us take this definition over to our non-commutative framework. Then

$$\mathcal{L} = \mathrm{d}x\,\tfrac{1}{2}\left[\sum_{i=1}^{n}\partial_i\phi(x)^2 + m^2\,\phi(x)^2\right] =: \mathrm{d}x\,L(x) \tag{5.10}$$

where $\mathrm{d}\boldsymbol{x} = \mathrm{d}x^1\cdots\mathrm{d}x^n$ is the volume form and ∂_i is the discrete derivative. The corresponding action for a volume $\mathcal{V}\subset\mathbb{R}^n$ is obtained by integrating $\mathcal{L}$ over $\mathcal{V}$

$$\mathcal{S} := \int_{\mathcal{V}}\mathrm{d}x\,L(x) \tag{5.11}$$

using the integral defined in appendix A. As shown there, this integral is well-defined only if the volume $\mathcal{V}$ is the union of n-dimensional rectangular cells with sides of length a^i, $i = 1,\ldots,n$. This amounts to specifying a lattice with spacings a^i and taking $\mathcal{V}$ to be the volume which fills the lattice. We should stress, however, that the lattice structure or, more precisely, the associated discrete translation group is already encoded in the differential calculus. The evaluation of the integral is now easily done using (A.11). We obtain

$$\begin{aligned}\mathcal{S}(\mathcal{V}) &= \tfrac{1}{2}\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_n=0}^{N_n-1}\int_{(k_1a^1,\ldots,k_na^n)}^{((k_1+1)a^1,\ldots,(k_n+1)a^n)}\mathrm{d}x\,L(x)\\ &= \tfrac{1}{2}\sum_{k_1=0}^{N_1-1}\cdots\sum_{k_n=0}^{N_n-1}a^1\cdots a^n\left\{\sum_{i=1}^{n}\frac{1}{(a^i)^2}\,[\phi(k_1a^1,\ldots,(k_i+1)a^i,\ldots,k_na^n)\right.\\ &\qquad\left.-\,\phi(k_1a^1,\ldots,k_na^n)]^2 + m^2\,\phi(k_1a^1,\ldots,k_na^n)^2\right\}.\end{aligned} \tag{5.12}$$

A corner of the lattice has been chosen as the origin of the coordinate system (cf figure 1). (5.12) is the usual lattice version of the action.

If $a^1 = a^2 = \cdots = a^n =: a$, the calculus generates a lattice like the one shown in figure 1 on the continuum over which we integrate. The action can then be rewritten in the form

$$S(\mathcal{V}) = \tfrac{1}{2}\,(a)^n \left(\sum_{\{k,\ell\}} \frac{1}{(a)^2}\,(\phi_k - \phi_\ell)^2 + \sum_k m^2\,\phi_k^2 \right) \tag{5.13}$$

where ϕ_k is the value of ϕ at the lattice site k and $\{k,\ell\}$ represents the set of all nearest neighbouring sites.

Remark. If $a^i \neq 0$, $i = 1, \ldots, n$, the closed 1-form

$$\vartheta := -\sum_i \frac{1}{a^i}\,\mathrm{d}x^i \tag{5.14}$$

satisfies

$$\mathrm{d}f = [\,\vartheta\,,\,f\,] \tag{5.15}$$

as a consequence of (the generalization of) (3.7). Moreover, (2.12) and (2.13) are satisfied. We therefore have a representation of d in the sense of the discussion in the last paragraph of section 2.

6. Gauge theory

In this section we generalize and extend the results of section 4 to the differential calculus (5.1) in n dimensions. The ordinary Yang–Mills Lagrangian formulated in terms of differential forms and the Hodge operator can be generalized to the non-commutative calculus. In this way we recover the correct lattice action for lattice gauge theory.

If $\mathbf{A} = \sum_i \mathrm{d}x^i\,A_i$ is a connection in the sense of section 4 we define

$$G_i(x) := \mathsf{I} - a^i\,A_i(x) \qquad \text{(no summation)}\,. \tag{6.1}$$

Then

$$G'_i(x) = U(x + a^i)\,G_i(x)\,U(x)^{-1} \tag{6.2}$$

under a gauge transformation with a group valued function U. The modified differential

$$Dx^i := \mathrm{d}x^i\,G_i(x) \qquad \text{(no summation)} \tag{6.3}$$

transforms covariantly as

$$D'x^i = U(x)\,Dx^i\,U(x)^{-1}\,. \tag{6.4}$$

If ψ is a field in the fundamental representation of the gauge group, a left and right covariant derivative of ψ are introduced by

$$\begin{aligned} D\psi &:= \mathrm{d}\psi + \mathsf{A}\,\psi \\ &= \sum_{i=1}^{n} \nabla_{-i}\psi\, \mathrm{d}x^i = \sum_{i=1}^{n} Dx^i\, \nabla_i \psi\,. \end{aligned} \tag{6.5}$$

One finds the following expressions for these covariant derivatives

$$\begin{aligned} (\nabla_i\psi)(x) &= G_i(x)^{-1}\,[\,(\partial_i\psi)(x) + A_i(x)\,\psi(x)\,] \\ &= \frac{1}{a^i}\,[\,G_i(x)^{-1}\,\psi(x+a^i) - \psi(x)\,] \end{aligned} \tag{6.6}$$

$$(\nabla_{-i}\psi)(x) = \frac{1}{a^i}\,[\,\psi(x) - G_i(x-a^i)\,\psi(x-a^i)\,]\,. \tag{6.7}$$

The field strength of A is

$$\begin{aligned} \mathsf{F} &:= \mathrm{d}\mathsf{A} + \mathsf{A}^2 \\ &= \tfrac{1}{2}\sum_{i,j} \mathrm{d}x^i\,\mathrm{d}x^j\,[\,(\partial_i A_j)(x) - (\partial_j A_i)(x) + A_i(x+a^j)\,A_j(x) \\ &\quad - A_j(x+a^i)\,A_i(x)\,] \\ &= \tfrac{1}{2}\sum_{i,j} \mathrm{d}x^i\,\mathrm{d}x^j\,\frac{1}{a^i a^j}\,[\,G_i(x+a^j)\,G_j(x) - G_j(x+a^i)\,G_i(x)\,] \\ &= \sum_{i,j} \mathrm{d}x^i\,\mathrm{d}x^j\,\frac{1}{a^i a^j}\,G_i(x+a^j)\,G_j(x) \\ &= \sum_{i,j} \frac{1}{a^i a^j}\,Dx^i\,Dx^j \end{aligned} \tag{6.8}$$

and transforms as

$$\mathsf{F}' = U(x)\,\mathsf{F}\,U(x)^{-1} \tag{6.9}$$

under a gauge transformation (4.2). In contrast to the differentials $\mathrm{d}x^i$ the covariant differentials Dx^i do not anticommute as a consequence of the non-commutativity between functions and differentials. Equation (6.8) relates the field strength F to the symmetric part of $Dx^i\,Dx^j$.

Let us now consider the Yang–Mills Lagrangian

$$\mathcal{L}_{\mathrm{YM}} := \mathrm{tr}\,[\,(\star\,\mathsf{F}^\dagger)\,\mathsf{F}\,] + \text{ cc} \tag{6.10}$$

for the gauge group $\mathrm{SU}(m)$ in n dimensions. For the undeformed differential calculus this is a familiar expression. With the generalized $\star$ operator introduced in section 6 it also makes sense in our non-commutative framework. Remembering that our $\star$ operator does not commute with functions, the reader may expect a strange transformation rule for $\mathcal{L}_{\rm YM}$. It is, however, gauge invariant as will be demonstrated in the following. From (6.8) we have

$$\mathsf{F}^\dagger = \sum_{i,j} \frac{1}{a^i a^j} \, dx^i \, dx^j \, G_j(x)^{-1} \, G_i(x+a^j)^{-1} \tag{6.11}$$

where we used

$$G_i(x)^\dagger = G_i(x)^{-1} \, . \tag{6.12}$$

Acting with $\star$ on (6.11) and using (5.8) yields

$$\star \, \mathsf{F}^\dagger = \sum_{i,j} \frac{1}{a^i a^j} \star (dx^i \, dx^j) \; G_j(x-a^i-a^j)^{-1} \, G_i(x-a^i)^{-1} \, . \tag{6.13}$$

Under a gauge transformation this transforms as

$$\star \, (\mathsf{F}'^\dagger) = U(x-\alpha) \, (\star \, \mathsf{F}^\dagger) \, U(x)^{-1} \tag{6.14}$$

where

$$\alpha := \sum_{k=1}^{n} a^k \, . \tag{6.15}$$

Hence

$$\begin{aligned} (\star \, \mathsf{F}'^\dagger) \, \mathsf{F}' &= U(x-\alpha) \, (\star \, \mathsf{F}^\dagger) \, \mathsf{F} \, U(x)^{-1} \\ &= dx \, U(x) \, \mathsf{L}(x) \, U(x)^{-1} \end{aligned} \tag{6.16}$$

where we have written

$$(\star \, \mathsf{F}^\dagger) \, \mathsf{F} = dx \, \mathsf{L}(x) \tag{6.17}$$

with a (matrix-valued) function $\mathsf{L}(x)$ and the volume form dx. This implies that $\mathcal{L}_{\rm YM}$ is indeed gauge invariant. Let us now determine $\mathsf{L}(x)$ explicitly

$$\begin{aligned} (\star \, \mathsf{F}^\dagger) \, \mathsf{F} = & \sum_{i,j,k,\ell} \frac{1}{a^i a^j a^k a^\ell} \star (dx^i \, dx^j) \, dx^k \, dx^\ell \\ & \times G_j(x-a^i-a^j+a^k+a^\ell)^{-1} \, G_i(x-a^i+a^k+a^\ell)^{-1} \, G_k(x+a^\ell) \, G_\ell(x) \\ = & \sum_{i,j} \frac{1}{(a^i a^j)^2} \, dx \, (\, G_j(x)^{-1} \, G_i(x+a^j)^{-1} \, G_i(x+a^j) \, G_j(x) \\ & - G_j(x)^{-1} \, G_i(x+a^j)^{-1} \, G_j(x+a^i) \, G_i(x) \,) \end{aligned} \tag{6.18}$$

where we have used the identity

$$\star \, (dx^i \, dx^j) \, dx^k \, dx^\ell = (\delta^{ik} \delta^{j\ell} - \delta^{i\ell} \delta^{jk}) \, dx \tag{6.19}$$

which follows from (B.3). Our final result is

$$(\star\mathsf{F}^\dagger)\,\mathsf{F} = \mathrm{d}\boldsymbol{x} \sum_{i,j} \frac{1}{(a^i a^j)^2} \left(\mathrm{I} - G_j(x)^{-1} G_i(x+a^j)^{-1} G_j(x+a^i)\, G_i(x) \right) \tag{6.20}$$

where I is the unit element of the group. By taking the trace this yields precisely the Lagrangian which has been proposed for lattice gauge theory (see [8,9] and references given there). The Lagrangian is actually defined on a space continuum. As in the case of the scalar field discussed in the previous section, the integration restricts it to a lattice. The last term in (6.20) then describes a Wilson loop around a plaquette. If x_α denotes a site of the lattice, $G_i(x_\alpha)$ associates a group element with the link between the neighbouring sites x_α and $x_\beta := x_\alpha + a^i$. In [14] a reduction of a lattice gauge theory to a matrix model has been considered by identifying the variables G_i on the links in the same direction. In our formalism this corresponds to the restriction to *constant* variables (since $\mathrm{d}f = 0$ for a function $f(x)$ means that f is periodic with periods a^i).

Remark. Let us introduce a gauge theoretical analogue of ϑ defined in (5.14)

$$\Theta := -\sum_i \frac{1}{a^i}\, Dx^i\,. \tag{6.21}$$

The exterior covariant derivative of Θ

$$\begin{aligned} D\Theta &:= -\sum_i \frac{1}{a^i} \left(\mathrm{d}Dx^i + \mathsf{A}\, Dx^i + Dx^i\, \mathsf{A} \right) \\ &= \mathrm{d}\mathsf{A} + 2\,\mathsf{A}^2 - \sum_i \frac{1}{a^i} \left(\mathsf{A}\, \mathrm{d}x^i + \mathrm{d}x^i\, \mathsf{A} \right) \\ &= 2\,\mathsf{F} \end{aligned} \tag{6.22}$$

is just the Yang–Mills field strength. (6.8) can also be written in the form $\mathsf{F} = \Theta^2$.

7. Conclusions

We think that our results impressively demonstrate the usefulness of non-commutative differential calculus in physics. We have shown that the deformation (5.1) of the classical differential calculus on the algebra of functions on a manifold transforms a continuum theory to a corresponding lattice theory. As examples the cases of a scalar field and an SU(m) gauge field in n dimensions have been treated. We have formulated a common framework which includes both continuum and lattice theories. They are merely distinguished by the vanishing or non-vanishing of the deformation parameters which appear in the commutation relations between functions and differentials. Since the lattice structure (coded in the deformation parameters) appears at the most basic and common mathematical level needed to formulate dynamics for physical fields and particles, our prescription for the passage from a continuum theory to a lattice theory is *universal*, i.e. does not depend on the particular theory.

A relation between the non-commutative differential calculus discussed in this work and q-calculus (which plays a role in particular in the context of quantum groups) has been established in [16].

Among the consistent differential calculi on function algebras we have now two examples of deformations of the standard differential calculus—the one presented in section 5 and the calculus of [11]—which are related to physics. There are further solutions of the consistency conditions. Some examples emerged from our classification of differential calculi with constant deformation parameters in two dimensions (appendices C and D). It would be nice to have a corresponding classification in higher dimensions. Exploring the features of further solutions may lead us to new surprises.

Appendix A. Integrals

In this appendix we define an integral for the differential calculus introduced in section 3 for one dimension and generalized to arbitrary dimensions in section 5. The basic property of the indefinite integral is

$$\int \mathrm{d}f = f \qquad (+\text{'constant' function}) \tag{A.1}$$

for an arbitrary function f. Let us first consider the one-dimensional case. Then, for example

$$\int \mathrm{d}x\, x = \int \mathrm{d}(x^2) - x\,\mathrm{d}x = x^2 - \int (\mathrm{d}x x + a\,\mathrm{d}x) \tag{A.2}$$

using (3.1). This implies

$$\int \mathrm{d}x\, x = \tfrac{1}{2}(x^2 - a\,x) \tag{A.3}$$

(plus an arbitrary periodic function). Using (3.4) and the binomial formula we have

$$\begin{aligned} x^{k+1} &= \int \mathrm{d}(x^{k+1}) = \int \mathrm{d}x\, \overrightarrow{\partial}\, x^{k+1} \\ &= \frac{1}{a}\int \mathrm{d}x\, \left[(x+a)^{k+1} - x^{k+1}\right] \\ &= \sum_{\ell=1}^{k+1} \binom{k+1}{\ell} a^{\ell-1} \int \mathrm{d}x\, x^{k+1-\ell} \end{aligned} \tag{A.4}$$

which leads to the formula

$$(k+1)\int \mathrm{d}x\, x^k = x^{k+1} - \sum_{\ell=2}^{k+1} \binom{k+1}{\ell} a^{\ell-1} \int \mathrm{d}x\, x^{k+1-\ell} \tag{A.5}$$

from which the integrals $\int \mathrm{d}x\, x^n$ can be calculated recursively. All these formulae have to be understood modulo addition of an arbitrary periodic function (with period a). Because of this reason a *definite* integral is not well defined, in general. The only exception is the case when the integration domain is a multiple of the period a.

Lemma.

$$\int_{x_0}^{x_0+a} \mathrm{d}x\, x^k = a\, x_0^k \qquad (k = 0, 1, 2, \ldots). \tag{A.6}$$

Proof. We use induction on k. The formula is easily verified for $k = 0, 1, 2$. Assuming that it holds for all powers of x smaller than a given k, we obtain from (A.5)

$$\begin{aligned}(k+1) \int_{x_0}^{x_0+a} \mathrm{d}x\, x^k &= (x_0 + a)^{k+1} - x_0^{k+1} - \sum_{\ell=2}^{k+1} \binom{k+1}{\ell} a^\ell\, x_0^{k+1-\ell} \\ &= (k+1)\, a\, x_0^k\end{aligned}$$

using the binomial expansion. □

As a consequence we have

$$\int_{x_0}^{x_0+a} \mathrm{d}x\, f(x) = a\, f(x_0) \tag{A.7}$$

for polynomials $f(x)$ (by linearity of the integral) and we can extend the definite integral taken over an interval of length a as a δ-distribution to arbitrary functions

$$\int_{x_0}^{x_0+a} \mathrm{d}x = a\, \delta_{x_0} \tag{A.8}$$

(see also [16] for a different proof).

In higher dimensions we define the integral for a function $f(x) = f(x^1, \ldots, x^n)$ which is of the form

$$f(x) = \sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n}\, f_{i_1}(x^1) \cdots f_{i_n}(x^n) \tag{A.9}$$

(with constant coefficients) by

$$\int \mathrm{d}x\, f(x) := \sum_{i_1,\ldots,i_n} c_{i_1\ldots i_n} \int \mathrm{d}x^1\, f_{i_1}(x^1) \cdots \int \mathrm{d}x^n\, f_{i_n}(x^n). \tag{A.10}$$

Using the one-dimensional result (A.7) we find

$$\int_{x_0}^{x_0+a} \mathrm{d}x\, f(x) = a^1 \cdots a^n\, f(x_0) = a^1 \cdots a^n\, \delta_{x_0} f \tag{A.11}$$

where $x_0 + a$ stands for $(x_0^1 + a^1, \ldots, x_0^n + a^n)$ and

$$\int_{x_0}^{x_0+a} \mathrm{d}x := \int_{x_0^1}^{x_0^1+a^1} \mathrm{d}x^1 \cdots \int_{x_0^n}^{x_0^n+a^n} \mathrm{d}x^n\,. \tag{A.12}$$

This definite integral extends as a δ-distribution to arbitrary functions $f(x)$.

Appendix B. The Laplace–Beltrami operator

In section 5 we introduced a Hodge $\star$ operator for the non-commutative differential calculus (5.1) in n dimensions†. This allows us to generalize the classical definition

$$\delta := \star \mathrm{d} \star (-1)^{n(r+1)} \tag{B.1}$$

(acting on r-forms) of the adjoint operator associated with d to the non-commutative framework. An r-form ω can be written as

$$\omega(x) = \frac{1}{r!} \sum \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r}\, \omega_{i_1 \ldots i_r}(x)\, . \tag{B.2}$$

Here and in the following summation is understood over all repeated indices. With the help of (5.8) and the identities

$$\begin{aligned} \mathrm{d}x^j \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k}) &= \delta^{j i_k} \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_{k-1}}) - \delta^{j i_{k-1}} \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_{k-2}} \mathrm{d}x^{i_k}) \\ &\quad + \cdots + (-1)^{k-1} \delta^{j i_1} \star (\mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_k}) \end{aligned} \tag{B.3}$$

$$\star \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k}) = (-1)^{k(n-k)} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_k} \tag{B.4}$$

we find

$$\begin{aligned} \mathrm{d} \star \mathrm{d} \star \omega(x) &= \mathrm{d} \star \mathrm{d} \frac{1}{r!} \sum \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r})\, \omega_{i_1 \ldots i_r}(x - a^{i_1} - \cdots - a^{i_r}) \\ &= \mathrm{d} \star \frac{1}{r!} \sum \mathrm{d}x^j \star (\mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r})\, \partial_j \omega_{i_1 \ldots i_r}(x - a^{i_1} - \cdots - a^{i_r}) \\ &= \frac{(-1)^{r-1}}{(r-1)!} \mathrm{d} \star \sum \star (\mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_r})\, \partial_{-j} \omega_{j i_2 \ldots i_r}(x - a^{i_2} - \cdots - a^{i_r}) \\ &= \frac{(-1)^{r-1}}{(r-1)!} \mathrm{d} \sum \star \star (\mathrm{d}x^{i_2} \cdots \mathrm{d}x^{i_r})\, \partial_{-j} \omega_{j i_2 \ldots i_r}(x - \alpha) \\ &= \frac{(-1)^{n(r-1)}}{(r-1)!} \sum \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r}\, \partial_{i_1} \partial_{-j} \omega_{j i_2 \ldots i_r}(x - \alpha) \end{aligned} \tag{B.5}$$

and similarly

$$\begin{aligned} \star \mathrm{d} \star \mathrm{d}\omega(x) = (-1)^{nr} \sum \Big(& \frac{1}{r!} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r}\, \partial_j \partial_{-j} \omega_{i_1 \ldots i_r}(x - \alpha) \\ & - \frac{1}{(r-1)!} \mathrm{d}x^{i_1} \cdots \mathrm{d}x^{i_r}\, \partial_{-j} \partial_{i_1} \omega_{j i_2 \ldots i_r}(x - \alpha) \Big) \end{aligned} \tag{B.6}$$

with α defined in (6.15). Using

$$[\,\partial_{-i}\, , \partial_j\,] = 0 \tag{B.7}$$

† The metric is taken to be Euclidean and has the components δ_{ij} with respect to the coordinates x^i. The following calculations are easily translated to the case of the Minkowski metric with components η_{ij} in the coordinate system x^i.

we obtain for the *Laplace–Beltrami operator*

$$\Delta := \mathrm{d}\,\delta + \delta\,\mathrm{d} \tag{B.8}$$

the following formula

$$\Delta\,\omega(x) = -\frac{1}{r!}\sum \mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_r}\,\partial_j\partial_{-j}\omega_{i_1\ldots i_r}(x-\alpha)\,. \tag{B.9}$$

Here we recover the lattice version of the Laplace–Beltrami operator (see [18], for example), apart from the shift $x \mapsto x - \sum_k a^k$ in the argument. It appears in the description of fermions on a lattice by the Dirac–Kähler equation (see [17,18], for example).

Appendix C. Classification of two-dimensional calculi

In this appendix we solve the consistency conditions for the class of differential calculi introduced in section 2 with the restriction to two dimensions and *real constant* coefficients $C^{ij}{}_k$. We classify the solutions into orbits with respect to $\mathrm{GL}(2,\mathbb{R})$-transformations which mix the coordinates x^i.

Every two-dimensional matrix can be written as a linear combination of the 2×2 unit matrix $\mathbf{I}$ and a trace-free matrix. Since an Abelian subalgebra of the algebra of 2×2 matrices is at most two-dimensional, the consistency condition $[\mathbf{C}^1,\mathbf{C}^2]=0$ implies

$$\mathbf{C}^i = \lambda^i\,\mathbf{I} + \mu^i\,\mathbf{T} \tag{C.1}$$

with $\mathrm{tr}\,\mathbf{T}=0$. Then

$$\lambda^i = \tfrac{1}{2}\,\mathrm{tr}\,\mathbf{C}^i\,. \tag{C.2}$$

Under a $\mathrm{GL}(2,\mathbb{R})$-transformation with matrix $A=(A^i_j)$ the matrices $\mathbf{C}^i$ transform as follows

$$\mathbf{C}'^i = A^i_j\,\left(A\,\mathbf{C}^j A^{-1}\right) = (A^i_j\,\lambda^j)\,\mathbf{I} + (A^i_j\,\mu^j)\,\left(A\,\mathbf{T}\,A^{-1}\right)\,. \tag{C.3}$$

Hence

$$\mathbf{C}'^i = \lambda'^i\,\mathbf{I} + \mu'^i\,\mathbf{T}' \tag{C.4}$$

with

$$\lambda'^i = A^i_j\,\lambda^j \qquad \mu'^i = A^i_j\,\mu^j \qquad \mathbf{T}' = A\,\mathbf{T}\,A^{-1}\,. \tag{C.5}$$

The next step is to choose representatives for $\mathbf{T}$ from every orbit of real trace-free matrices with respect to the adjoint action of $\mathrm{GL}(2,\mathbb{R})$ which is the same as the adjoint action of $SL(2,\mathbb{R})$. The orbits are easily determined and we discuss convenient representatives in turn. For each of them we have to solve the consistency condition (2.6) which means that the second row of $\mathbf{C}^1$ is equal to the first row of $\mathbf{C}^2$.

(1) $\mathbf{T} = 0$. The consistency condition (2.6) enforces $\lambda^i = 0$ and therefore $\mathbf{C}^i = 0$ which is the classical (undeformed) differential calculus.
(2) Two other orbits are represented by

$$\mathbf{T} = \pm \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{C.6}$$

It is sufficient to treat the case with the positive sign since the sign can be absorbed into μ^i. (2.6) requires $\mu^1 = \lambda^2$, $\lambda^1 = 0$ so that

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ \lambda^2 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} \lambda^2 & 0 \\ \mu^2 & \lambda^2 \end{pmatrix} .$$

The isotropy group of $\mathbf{T}$ is

$$G_T = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \middle| a, b \in \mathbb{R},\ a \neq 0 \right\} .$$

(a) If $\lambda^2 = 0$ (and $\mu^2 \neq 0$) we can use a G_T-transformation (with suitably chosen a) to achieve $\mu^2 = 1$. Hence

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{C.7}$$

(b) If $\lambda^2 \neq 0$ we can use the isotropy group to arrange for $\lambda^2 = 1$. This breaks G_T to the subgroup of matrices

$$\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \qquad (\, b \in \mathbb{R}\,) .$$

For these matrices and the $\mathbf{C}^i$ given above the transformation law (C.4) takes the simple form

$$\mathbf{C}'^1 = \mathbf{C}^1 \qquad \mathbf{C}'^2 = b\,\mathbf{C}^1 + \mathbf{C}^2 \tag{C.8}$$

which shows that we can choose b to eliminate μ^2. We arrive at

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{C.9}$$

(3) Another family of orbits is represented by

$$\mathbf{T} = \beta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad (\beta \in \mathbb{R},\ \beta \neq 0) . \tag{C.10}$$

It is sufficient to consider the case $\beta = 1$ since β can be absorbed into μ^i. (2.6) now leads to $\mu^1 = \lambda^1$, $\mu^2 = -\lambda^2$. The isotropy group of $\mathbf{T}$ is in the present case

$$G_T = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \middle| a, b \in \mathbb{R},\ a \neq 0\,,\ b \neq 0 \right\} .$$

(C.4) with $A \in G_T$ becomes

$$\mathbf{C}'^1 = a\,\lambda^1\,(\mathbf{1}+\mathbf{T}) \qquad \mathbf{C}'^2 = b\,\lambda^2\,(\mathbf{1}-\mathbf{T}) \tag{C.11}$$

so that we can transform $2\lambda^i$ to 1 if $\lambda^i \neq 0$. We thus obtain the following two solutions

$$\mathbf{C}^1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{C.12}$$

and

$$\mathbf{C}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} . \tag{C.13}$$

(4) The remaining family of orbits is represented by

$$\mathbf{T} = \beta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad (\beta \in \mathbb{R},\, \beta \neq 0) . \tag{C.14}$$

Again, we only have to consider $\beta = 1$. (2.6) leads to

$$\mathbf{C}^1 = \begin{pmatrix} \lambda^1 & -\lambda^2 \\ \lambda^2 & \lambda^1 \end{pmatrix} \qquad \mathbf{C}^2 = \mathbf{T}\,\mathbf{C}^1 . \tag{C.15}$$

It is more convenient to express $\mathbf{C}^1$ in the form

$$\mathbf{C}^1 = \rho \begin{pmatrix} \cos\chi & -\sin\chi \\ \sin\chi & \cos\chi \end{pmatrix} \tag{C.16}$$

with $\rho, \chi \in \mathbb{R}$. The isotropy group of $\mathbf{T}$ is

$$G_T = \left\{ a \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} \middle|\, a, \varphi \in \mathbb{R},\, a \neq 0 \right\}$$

and can be used to transform ρ to 1. Under the remaining G_T-transformations with $a = 1$ the transformation law (C.4) for our matrix $\mathbf{C}^1$ becomes

$$\mathbf{C}'^1 = (\cos\varphi\,\mathbf{1} - \sin\varphi\,\mathbf{T})\,\mathbf{C}^1 \tag{C.17}$$

so that

$$\mathbf{C}'^1 = \begin{pmatrix} \cos(\varphi+\chi) & -\sin(\varphi+\chi) \\ \sin(\varphi+\chi) & \cos(\varphi+\chi) \end{pmatrix} . \tag{C.18}$$

Choosing $\varphi = -\chi$ we end up with

$$\mathbf{C}^1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \mathbf{C}^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{C.19}$$

Let us summarize our results.

Theorem. All consistent differential calculi satisfying the commutation relations (2.3) with real constant coefficients $C^{ij}{}_k$ in two dimensions are given modulo GL(2, $\mathbb{R}$)-transformations by (C.7), (C.9), (C.12), (C.13), (C.19) and the standard (undeformed) calculus.

The classification is the same for purely imaginary coefficients $C^{ij}{}_k$. The corresponding representatives of orbits are simply obtained by multiplication of the $C^{ij}{}_k$ found above with the imaginary unit i. Since the requirement of having an operation * on the differential algebra which generalizes the conjugation of complex numbers (see section 2) restricts complex coefficients $C^{ij}{}_k$ to be either real or imaginary, we thus have a complete classification in two dimensions.

(C.7) is the two-dimensional version of the calculus considered in [11]. (C.12) and (C.13) are the direct sum of one-dimensional calculi (see section 3). (C.9) and (C.19) determine new differential calculi and we refer to appendix D for a brief discussion.

Appendix D. Additional differential calculi in two dimensions

In appendix C we have determined and classified all consistent differential calculi of the form (2.2) and (2.3) with real or imaginary constant coefficients $C^{ij}{}_k$ in two dimensions. Besides direct sums of the one-dimensional calculus of section 3 there are modulo GL(2, $\mathbb{R}$)-transformations three additional possibilities which we discuss in the following.

(1) The first is given by

$$\begin{aligned} [\, x^1 \,, \mathrm{d}x^1 \,] &= 0 \\ [\, x^1 \,, \mathrm{d}x^2 \,] &= 0 \\ [\, x^2 \,, \mathrm{d}x^1 \,] &= 0 \\ [\, x^2 \,, \mathrm{d}x^2 \,] &= q \, \mathrm{d}x^1 \end{aligned} \tag{D.1}$$

where q is a complex† parameter. For $q = 1$ this corresponds to (C.7). This is the two-dimensional version of the differential calculus studied in [11]. For this calculus there is no 1-form ϑ to express d as the (anti-) commutator with ϑ (see the last paragraph of section 2).

If we define *left* and *right partial derivatives* by

$$\mathrm{d}f = \partial_{-i} f \, \mathrm{d}x^i = \mathrm{d}x^i \, \partial_i f \tag{D.2}$$

then

$$\partial_{-1} = \frac{\partial}{\partial x^1} - \frac{q}{2} \left(\frac{\partial}{\partial x^2} \right)^2 \qquad \partial_{-2} = \frac{\partial}{\partial x^2} \tag{D.3}$$

† The two possibilities to extend the complex conjugation to the whole differential algebra discussed in section 2 constrain q to be real or imaginary.

and

$$\partial_1 = \frac{\partial}{\partial x^1} + \frac{q}{2}\left(\frac{\partial}{\partial x^2}\right)^2 \qquad \partial_2 = \frac{\partial}{\partial x^2} \tag{D.4}$$

where $\partial/\partial x^i$ are ordinary partial derivatives.
(2) The second calculus is

$$\begin{aligned} [x^1, dx^1] &= 0 \\ [x^1, dx^2] &= q\, dx^1 \\ [x^2, dx^1] &= q\, dx^1 \\ [x^2, dx^2] &= q\, dx^2 \end{aligned} \tag{D.5}$$

and corresponds to (C.9) for $q = 1$. The exterior derivative d can be expressed in the form (2.12) with the 1-form

$$\vartheta = -\frac{1}{q}\, dx^2 \,. \tag{D.6}$$

The partial derivatives are realized by

$$\partial_{-1} = \exp\left(-q\frac{\partial}{\partial x^2}\right)\frac{\partial}{\partial x^1} \qquad \partial_{-2} = \frac{1}{q}\left[1 - \exp\left(-q\frac{\partial}{\partial x^2}\right)\right] \tag{D.7}$$

and

$$\partial_1 = \exp\left(q\frac{\partial}{\partial x^2}\right)\frac{\partial}{\partial x^1} \qquad \partial_2 = \frac{1}{q}\left[\exp\left(q\frac{\partial}{\partial x^2}\right) - 1\right]. \tag{D.8}$$

Their action on a function $f(x^1, x^2)$ is given by

$$(\partial_{-1}f)(x^1, x^2) = \frac{\partial f}{\partial x^1}(x^1, x^2 - q) \tag{D.9}$$

$$(\partial_{-2}f)(x^1, x^2) = \frac{1}{q}\left[f(x^1, x^2) - f(x^1, x^2 - q)\right] \tag{D.10}$$

and

$$(\partial_1 f)(x^1, x^2) = \frac{\partial f}{\partial x^1}(x^1, x^2 + q) \tag{D.11}$$

$$(\partial_2 f)(x^1, x^2) = \frac{1}{q}\left[f(x^1, x^2 + q) - f(x^1, x^2)\right]\,. \tag{D.12}$$

(3) The third calculus is

$$\begin{aligned} [x^1, dx^1] &= q\, dx^1 \\ [x^1, dx^2] &= q\, dx^2 \\ [x^2, dx^1] &= q\, dx^2 \\ [x^2, dx^2] &= -q\, dx^1\,. \end{aligned} \tag{D.13}$$

When $q = 1$ this corresponds to (C.19). The exterior derivative d can be expressed in the form (2.12) with the 1-form

$$\vartheta = -\frac{1}{q}\,\mathrm{d}x^1\,. \tag{D.14}$$

The left and right partial derivatives are realized by

$$(\partial_{-1}f)(x^1,x^2) = \frac{1}{q}\left[f(x^1,x^2) - \left(\cos\left(q\,\frac{\partial}{\partial x^2}\right)f\right)(x^1-q,x^2)\right] \tag{D.15}$$

$$(\partial_{-2}f)(x^1,x^2) = \frac{1}{q}\left(\sin\left(q\,\frac{\partial}{\partial x^2}\right)f\right)(x^1-q,x^2) \tag{D.16}$$

and

$$(\partial_1 f)(x^1,x^2) = \frac{1}{q}\left[\left(\cos\left(q\,\frac{\partial}{\partial x^2}\right)f\right)(x^1+q,x^2) - f(x^1,x^2)\right] \tag{D.17}$$

$$(\partial_2 f)(x^1,x^2) = \frac{1}{q}\left(\sin\left(q\,\frac{\partial}{\partial x^2}\right)f\right)(x^1+q,x^2) \tag{D.18}$$

respectively. Introducing $z := (x^1 + \mathrm{i}\,x^2)/2$, the above commutation relations can be written as

$$\begin{aligned}
[\,z\,,\,\mathrm{d}z\,] &= q\,\mathrm{d}z \\
[\,z^*\,,\,\mathrm{d}z\,] &= 0 \\
[\,z\,,\,\mathrm{d}z^*\,] &= 0 \\
[\,z^*\,,\,\mathrm{d}z^*\,] &= q^*\,\mathrm{d}z^*
\end{aligned} \tag{D.19}$$

where the * denotes complex conjugation. This calculus thus emerges from the sum of two complex one-dimensional calculi

$$[\,z^i\,,\,\mathrm{d}z^j\,] = q_i\,\delta^{ij}\,\mathrm{d}z^j \qquad \text{(no summation)} \tag{D.20}$$

by imposing the constraints $(z^1)^* = z^2$, $q_1^* = q_2$. Each of the one-dimensional calculi is a complex version of the real calculus considered in section 3. The partial derivatives obtained by writing $\mathrm{d}f$ as a linear combination of $\mathrm{d}z$ and $\mathrm{d}z^*$ are discrete derivatives.

It still has to be seen whether the last two calculi listed above have applications to physics. Relaxing the assumption that the $C^{ij}{}_k$ in (2.3) are constants and replacing them by functions of the x^i will lead to further consistent differential calculi. The question whether these can be transformed to calculi with constant coefficients is a complicated problem (see the first remark in section 3).

References

[1] Connes A 1986 Non-commutative differential geometry *Publ. I.H.E.S.* **62** 257; 1990 *Géométrie non commutative* (Paris: InterEditions)
Kastler D 1988 *Cyclic Cohomology* (Paris: Hermann)
Coquereaux R 1990 Noncommutative geometry and theoretical physics *J. Geom. Phys.* **6** 425

[2] Majid S 1988 Hopf algebras for physics at the Planck scale *Class. Quantum Grav.* **5** 1587

[3] Dubois-Violette M, Kerner R and Madore J 1989 Classical bosons in a noncommutative geometry *Class. Quantum Grav.* **6** 1709

[4] Madore J 1992 The fuzzy sphere *Class. Quantum Grav.* **9** 69; 1991 Quantum mechanics on a fuzzy sphere *Phys. Lett.* **263B** 245; 1992 Fuzzy physics *Preprint* LPTHE Orsay 92/01

[5] Wilson K G 1974 Confinement of quarks *Phys. Rev.* D **10** 2445

[6] Drouffe J M and Itzykson C 1978 Lattice gauge fields *Phys. Rep.* **38** 134

[7] Guth A H 1980 Existence proof of a nonconfining phase in four-dimensional U(1) lattice gauge theory *Phys. Rev.* D **21** 2291

[8] Creutz M 1983 *Quarks, gluons and lattices* (Cambridge: Cambridge University Press)

[9] Kogut J B 1983 Lattice gauge theory approach to quantum chromodynamics *Rev. Mod. Phys.* **55** 785

[10] Woronowicz S L 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Commun. Math. Phys.* **122** 125

[11] Dimakis A and Müller-Hoissen F 1992 Noncommutative differential calculus, gauge theory and gravitation *Preprint* Göttingen GOET-TP 33/92

[12] Manin Yu I 1989 Multiparametric quantum deformation of the general linear supergroup *Commun. Math. Phys.* **123** 163

[13] Dimakis A and Müller-Hoissen F 1992 Quantum mechanics as noncommutative symplectic geometry *J. Phys. A: Math. Gen.* **25** 5625

[14] Eguchi T and Kawai H 1982 Reduction of dynamical degrees of freedom in the large-N gauge theory *Phys. Rev. Lett.* **48** 1063

[15] Parisi G 1982 A simple expression for planar field theories *Phys. Lett.* **112B** 463
Bhanot G, Heller U M and Neuberger H 1982 The quenched Eguchi–Kawai model *Phys. Lett.* **113B** 47
Eguchi T and Nakayama R 1983 Simplification of quenching procedures for large N spin models *Phys. Lett.* **122B** 59
Fabricius K, Haan O and Filk T 1984 Generalized twisted Eguchi–Kawai models *Phys. Lett.* **144B** 240

[16] Dimakis A and Müller-Hoissen F 1992 Quantum mechanics on a lattice and q-deformations *Phys. Lett.* **295B** 242

[17] Becher P and Joos H 1982 The Dirac–Kähler equation and fermions on the lattice *Z. Phys.* C **15** 343; 1984 On Kähler's geometric description of Dirac fields *Progress in Gauge Field Theory* ed G t'Hooft *et al* (New York: Plenum) p 247

[18] Rabin J M 1982 Homology theory of lattice fermion doubling *Nucl. Phys.* B **201** 315